Recitation 7. April 27

Focus: algebraic & geometric multiplicity, Jordan normal form, complex eigenvalues & eigenvectors, symmetric matrices

Given an $n \times n$ matrix A and an eigenvalue λ, then its:

$$\boxed{\textbf{algebraic multiplicity}} = \text{multiplicity of } \lambda \text{ as a root of the characteristic polynomial}$$

$$\boxed{\textbf{geometric multiplicity}} = \text{dimension of } N(A - \lambda I)$$

In general, we have the following inequality for all eigenvalues of A:

$$\boxed{\text{algebraic multiplicity} \geq \text{geometric multiplicity}}$$

If the inequality above is an equality for all eigenvalues λ of A, then $\mathbb{R}^n$ has a basis consisting only of eigenvectors, hence we can diagonalize the matrix A:

$$\boxed{A = VDV^{-1}}$$

where V is the matrix whose columns are eigenvectors, and D is the diagonal matrix with the eigenvalues on the diagonal. Even if the matrix A is not diagonalizable, we can always write it as:

$$A = V \left[\begin{array}{c|c|c|c} J_1 & 0 & \dots & 0 \\ \hline 0 & J_2 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & J_t \end{array} \right] V^{-1}$$

where each J_i is a $\boxed{\textbf{Jordan block}}$ of the form:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

As the eigenvalues are roots of the characteristic polynomial, quite often it turns out that they are $\boxed{\textbf{complex numbers}}$:

$$\boxed{z = a + bi}$$

where $a, b \in \mathbb{R}$ and i is a symbol that satisfies the relation $i^2 = -1$. Recall the following notions:

$$\boxed{\textbf{complex conjugate}} : \overline{a + bi} = a - bi$$

$$\boxed{\textbf{absolute value}} : |a + bi| = \sqrt{a^2 + b^2}$$

You may go back and forth between $\boxed{\textbf{Cartesian coordinates}}$ a, b and $\boxed{\textbf{polar coordinates}}$ r, θ:

$$\begin{cases} r = \sqrt{a^2 + b^2} \\ \theta = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) \end{cases} \quad \Leftrightarrow \quad \begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$

So with this in mind, we get the formula:

$$\boxed{a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

Symmetric $n \times n$ matrices are always diagonalizable with real eigenvalues and orthonormal eigenvectors:

$$\boxed{S = QDQ^{-1} = QDQ^T}$$

1. Which of the following matrices are diagonalizable? What are the algebraic and geometric multiplicities of the eigenvalues?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution: The characteristic polynomial of A is $(1-\lambda)(3-\lambda) - 1 \cdot 0 = (1-\lambda)(3-\lambda)$. The eigenvalues are 1 and 3, each with algebraic multiplicity 1. Since geometric multiplicities are at least 1 and at most the algebraic multiplicities, we learn that the geometric multiplicities are also 1 and A is diagonalizable.

The characteristic polynomial of B is $(3-\lambda)^2$. The sole eigenvalue is 3, with algebraic multiplicity 2. Since $B - 3I$ is the 0 matrix, its nullspace is 2-dimensional, so the eigenvalue 3 has geometric multiplicity 2. It follows that B is diagonalizable (in fact, it is diagonal).

The characteristic polynomial of C is $(2-\lambda)^2$. The sole eigenvalue is 2, with algebraic multiplicity 2. The matrix $C - 2I$ has rank 1 and nullspace of dimension 1. Thus, the eigenvalue 2 has geometric multiplicity 1, and C is *not* diagonalizable. In the language of the lecture notes, C is a *Jordan block*.

2. Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

and calculate $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$.

Solution: As we have seen, to compute the powers of a matrix, the easiest way is to diagonalize it:

$$\text{if } A = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1} \qquad \text{then } A^n = V \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} V^{-1}$$

If this were the case, we would have $e^A = V \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} V^{-1}$, quite a nice formula.

However, things are not so simple in this case, since the matrix A is not diagonalizable. Indeed, its characteristic polynomial is:

$$p(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - (-1) = \lambda^2$$

so the only eigenvalue is 0, with multiplicity 2. Since A is not the 0 matrix, this eigenvalue can only have multiplicity 1, hence A is not diagonalizable.

However, not all is lost. We can still put A in Jordan normal form. First we compute an eigenvector:

$$(A - 0I)\boldsymbol{v}_1 = 0 \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then we compute a vector $\boldsymbol{v}_2$ such that:

$$(A - 0I)\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \overset{\text{RREF}}{\Rightarrow} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So we may choose $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then if we consider the matrix:

$$V = \left[\begin{array}{c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{array} \right] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

we have:

$$A = VJV^{-1} \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

As before, we have $e^A = Ve^JV^{-1}$. However, e^J is easy to compute because $J^2 = 0$, hence:

$$e^J = I + J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We conclude that $e^A = V \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} V^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

3. Consider the matrix:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find its eigenvalues and eigenvectors. How many eigenvalues are complex? Are there complex conjugate pairs?

Solution: Considering the characteristic polynomial of X, we get:

$$p(\lambda) = \det(X - \lambda I) = 1 - \lambda^3$$

(you can compute the latter in many ways, but cofactor expansion is particularly simple, since the matrix has many zeroes). This polynomial factors as:

$$p(\lambda) = (1 - \lambda)(\lambda^2 + \lambda + 1) = (1 - \lambda)\left(\lambda - \frac{-1 + \sqrt{3}i}{2}\right)\left(\lambda - \frac{-1 - \sqrt{3}i}{2}\right)$$

Hence there is a real eigenvalue, namely 1, and two complex conjugate eigenvalues $\frac{-1 \pm \sqrt{3}i}{2}$.

The real eigenvalue 1 has eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which is easy to see.

The other two eigenvalues are the roots of unity of order 3, namely $\omega = e^{\frac{2\pi i}{3}}$ and $\omega^2 = e^{\frac{4\pi i}{3}}$. Their eigenvectors are:

$$\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}$$

respectively.

4. Consider the matrix:

$$Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Can you find an easy eigenvalue without computing the characteristic polynomial?
- Compute all eigenvectors for the above easy eigenvalue
- Can you use this to determine the remaining eigenvector and eigenvalue?

Solution: Note that the matrix Y is singular, hence it has an eigenvalue 0. Further note all the columns are the same, so 0 has a 2-dimensional vector space of eigenvectors, with basis given by:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(basically, the subspace of eigenvectors for the eigenvalue 0 are all 3-component vectors with the sum of the entries 0).

We have one more eigenvalue. To compute it, recall that the sum of the eigenvalues of a matrix is equal to its trace, and clearly $\mathrm{Tr}\ Y = 3$. This implies that the third eigenvalue is 3, since the other eigenvalue is 0 with multiplicity two.

As for the eigenvector corresponding to 3, we note that it has to be orthogonal to the eigenvectors corresponding to 0 (since the matrix is symmetric). Since the latter span the two-dimensional subspace consisting of vectors with the sum of entries equal to 0, the only possibility is for the eigenvector corresponding to 3 to be:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$